Math 10B with Professor Stankova
Worksheet, Discussion #26; Thursday, 4/26/2018
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1 Matrices and Eigenvalue Review

1.1 Example

1. Find the general solution to

$$\begin{cases} y_1'(t) = 3y_1(t) + 5y_2(t) \\ y_2'(t) = -y_1(t) + y_2(t) \end{cases}$$

Solution: The matrix associated to this is $A = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}$ and we find the eigenvalues which are when $\det(A - \lambda I) = \lambda^2 - 4\lambda + 4 = 0$. But this is when $\lambda = 2 \pm 2i$. The eigenvectors are gotten by looking at $A - \lambda I$. For $\lambda = 2 + 2i$, we get $\begin{pmatrix} 1-2i & 5 \\ -1 & -1-2i \end{pmatrix}$ and so we get an eigenvector of $\begin{pmatrix} 5 \\ 2i-1 \end{pmatrix}$. Thus one solution is $e^{\lambda t}\vec{v} = e^{(2+2i)t}\begin{pmatrix} 5 \\ 2i-1 \end{pmatrix}$. We use Euler's formula to write $e^{2t+2it} = e^{2t} \cdot e^{2it} = e^{2t}(\cos(2t) + i\sin(2t))$. So the solution is $e^{2t}\begin{pmatrix} 5\cos(2t) + 5i\sin(2t) \\ 2i\cos(2t) - \cos(2t) - 2\sin(2t) - i\sin(2t) \end{pmatrix}$. The real solution is $e^{2t}\begin{pmatrix} 5\cos(2t) \\ -\cos(2t) - 2\sin(2t) \end{pmatrix}$ and the imaginary solution is $e^{2t}\begin{pmatrix} 5\sin(2t) \\ 2\cos(2t) - \sin(2t) \end{pmatrix}$. Thus, the general solution is

$$c_1 e^{2t}\begin{pmatrix} 5\cos(2t) \\ -\cos(2t) - 2\sin(2t) \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 5\sin(2t) \\ 2\cos(2t) - \sin(2t) \end{pmatrix}.$$

2. Represent the second order linear homogeneous differential equation $y'' - 4y' + 3y = 0$ as a system of linear differential equations using $y_1(t) = y$ and $y_2(t) = y'(t)$.

Solution:

$$\begin{cases} y_1'(t) = y'(t) = y_2(t) \\ y_2'(t) = y''(t) = 4y'(t) - 3y(t) = 4y_2(t) - 3y_1(t) \end{cases}.$$

1.2 Problems

3. **TRUE** False If $A\vec{v} = 3\vec{v}$, then $A^{100}\vec{v} = 3^{100}\vec{v}$.

4. **TRUE** False When we write a second order homogeneous linear differential equation as a system of first order equations and solve for y_1, y_2, then whatever we get for y_2 will always be the derivative of whatever we get for y_1.

5. **TRUE** False If $\vec{x}, \vec{y}$ are two solutions to $\vec{z}' = A\vec{z}$, then any linear combination of them is also a solution.

6. **TRUE** False If λ is an eigenvalue for A, then there are infinitely many solutions to $A\vec{v} = \lambda\vec{v}$.

7. Find the general solution to

$$\begin{cases} y_1'(t) = 2y_1(t) - 2y_2(t) \\ y_2'(t) = y_1(t) + 4y_2(t) \end{cases}$$

Solution: The matrix is $\begin{pmatrix} 2 & -2 \\ 1 & 4 \end{pmatrix}$. The characteristic equation is $\lambda^2 - 6\lambda + 10 = 0$ which has roots $\lambda = 3 \pm i$. If we take $\lambda = 3 + i$, then the matrix $A - \lambda I$ is $\begin{pmatrix} -1-i & -2 \\ 1 & 1-i \end{pmatrix}$ and an eigenvector is $\begin{pmatrix} -2 \\ 1+i \end{pmatrix}$. Thus a complex solution is

$$e^{(3+i)t}\begin{pmatrix} -2 \\ 1+i \end{pmatrix} = \begin{pmatrix} -2(e^{3t}(\cos(t) + i\sin(t))) \\ (1+i)(e^{3t}(\cos(t) + i\sin(t))) \end{pmatrix} = \begin{pmatrix} -2e^{3t}\cos(t) - 2ie^{3t}\sin(t) \\ e^{3t}\cos(t) - e^{3t}\sin(t) + i(e^{3t}\cos(t) + e^{3t}\sin(t)) \end{pmatrix}$$

The general solution is a linear combination of the real and imaginary parts of the solution and hence it is

$$c_1 e^{3t}\begin{pmatrix} -2\cos(t) \\ \cos(t) - \sin(t) \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} -2\sin(t) \\ \cos(t) + \sin(t) \end{pmatrix}.$$

8. Find the general solution to

$$\begin{cases} y_1'(t) = y_1(t) - 2y_2(t) \\ y_2'(t) = 2y_1(t) + y_2(t) \end{cases} .$$

Solution: The matrix is $\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$. The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$ which has roots $\lambda = 1 \pm 2i$. If we take $\lambda = 1 + 2i$, then the matrix $A - \lambda I$ is $\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix}$ and an eigenvector is $\begin{pmatrix} -2 \\ 2i \end{pmatrix}$. Thus a complex solution is

$$e^{(1+2i)t}\begin{pmatrix} -2 \\ 2i \end{pmatrix} = \begin{pmatrix} -2(e^t(\cos(2t) + i\sin(2t))) \\ 2i(e^t(\cos(2t) + i\sin(2t))) \end{pmatrix} = \begin{pmatrix} -2e^t\cos(2t) - 2ie^t\sin(2t) \\ -2e^t\sin(2t) + 2ie^t\cos(2t) \end{pmatrix}.$$

The general solution is a linear combination of the real and imaginary parts of the solution and hence it is

$$c_1e^t\begin{pmatrix} -2\cos(2t) \\ -2\sin(2t) \end{pmatrix} + c_2e^t\begin{pmatrix} -2\sin(t) \\ 2\cos(2t) \end{pmatrix}.$$

9. Find the characteristic equation of the matrix of a system of linear differential equations if one solution is $\begin{pmatrix} 2e^{2t}\cos(2t) \\ 4e^{2t}\cos(2t) - 3e^{2t}\sin(2t) \end{pmatrix}$

Solution: Since you have e^{2t} and $\cos(2t)$, this tells us that the roots are $\lambda = 2 \pm 2i$. Therefore, the characteristic equation is $(\lambda - (2+2i))(\lambda - (2-2i)) = \lambda^2 - 4\lambda + 8 = 0$.

10. What are the eigenvalues and eigenvectors of the matrix of a system of linear differential equations if one solution is $\begin{pmatrix} 3e^{2t} + 5e^{4t} \\ e^{2t} - e^{4t} \end{pmatrix}$.

Solution: The solution is of the form $e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $e^{4t}\begin{pmatrix} 5 \\ -1 \end{pmatrix}$. Therefore, there is an eigenvalue of 2 with eigenvector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and there is an eigenvalue of 4 with eigenvector $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$.

11. Find the general solution to $y'' + 3y' - 10y = 0$ by writing it as a system of first order differential equations.

Solution: Let $y_1(t) = y(t)$ and $y_2(t) = y'(t)$. We represent it as

$$\begin{cases} y_1' = y_2 \\ y_2' = -3y_2 + 10y_1 \end{cases} .$$

The matrix is $\begin{pmatrix} 0 & 1 \\ 10 & -3 \end{pmatrix}$. The characteristic equation is $\lambda^2+3\lambda-10 = (\lambda+5)(\lambda-2) = 0$ which has roots $\lambda = -5, 2$. If we take $\lambda = -5$, then the matrix $A - \lambda I$ is $\begin{pmatrix} 5 & 1 \\ 10 & 2 \end{pmatrix}$ and an eigenvector is $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$. If we take $\lambda = 2$, then the matrix $A - \lambda I$ is $\begin{pmatrix} -2 & 1 \\ 10 & -5 \end{pmatrix}$ and an eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Thus the general solution is

$$c_1 e^{-5t} \begin{pmatrix} 1 \\ -5 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Thus, the solution is $y(t) = y_1(t) = c_1 e^{-5t} + c_2 e^{2t}$.

12. Find the general solution to $y'' + y' - 12y = 0$ by writing it as a system of first order differential equations.

Solution: Let $y_1(t) = y(t)$ and $y_2(t) = y'(t)$. We represent it as

$$\begin{cases} y_1' = y_2 \\ y_2' = -y_2 + 12y_1 \end{cases} .$$

The matrix is $\begin{pmatrix} 0 & 1 \\ 12 & -1 \end{pmatrix}$. The characteristic equation is $\lambda^2+\lambda-12 = (\lambda+4)(\lambda-3) = 0$ which has roots $\lambda = -4, 3$. If we take $\lambda = -4$, then the matrix $A - \lambda I$ is $\begin{pmatrix} 4 & 1 \\ 12 & 3 \end{pmatrix}$ and an eigenvector is $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$. If we take $\lambda = 3$, then the matrix $A - \lambda I$ is $\begin{pmatrix} -3 & 1 \\ 12 & -4 \end{pmatrix}$ and an eigenvector is $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Thus the general solution is

$$c_1 e^{-4t} \begin{pmatrix} 1 \\ -4 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Thus, the solution is $y(t) = y_1(t) = c_1 e^{-4t} + c_2 e^{3t}$.

13. Name 3 reasons why we care about eigenvalues and eigenvectors.

Solution:

- They allow us to diagonalize a matrix
- They allow us to solve a system of linear first order differential equations
- They allow us to solve a second order linear homogeneous differential equations
- They allow us to find solutions to recurrence relations

2 Linear Regression

2.1 Concepts

14. Often when given data points, we want to find the line of best fit through them. To them, we want to approximate them with a line $y = ax + b$. We represent this as a solution where we want to solve for a, b. In matrix vector form and data points (x_i, y_i), this is represented as

$$A\vec{x} = \vec{b} \to \begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

Often, we cannot find a perfect fit (if not all the points lie on the same line). So we want to find the error. One way to find the error is to take the least square error or $E = \sum (y_i - (ax_i + b))^2$, the sum of the squares of the error. The choice of a, b that minimizes this is

$$\begin{pmatrix} a \\ b \end{pmatrix} = (A^T A)^{-1} A^T \vec{b}.$$

2.2 Example

15. The number of people applying to Berkeley is given in the following table:

Year	2011	2012	2013	2014	2015	2016	2017
Applicants(in 1000s)	53	62	68	74	79	83	85

Predict how many people will apply this year.

Solution: We first write it in matrix form as

$$\begin{pmatrix} 2011 & 1 \\ 2012 & 1 \\ 2013 & 1 \\ 2014 & 1 \\ 2015 & 1 \\ 2016 & 1 \\ 2017 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 53 \\ 62 \\ 68 \\ 74 \\ 79 \\ 83 \\ 85 \end{pmatrix}.$$

Then we calculate $A^TA = \begin{pmatrix} 28393400 & 14098 \\ 14098 & 7 \end{pmatrix}$ and $A^T\vec{b} = \begin{pmatrix} 1015205 \\ 504 \end{pmatrix}$. Then

$$\begin{pmatrix} a \\ b \end{pmatrix} = (A^TA)^{-1}A^T\vec{b} = \frac{1}{28393400 \cdot 7 - 14098^2} \begin{pmatrix} 7 & -14098 \\ -14098 & 28393400 \end{pmatrix} \begin{pmatrix} 1015205 \\ 504 \end{pmatrix} \approx \begin{pmatrix} 5 \\ -10645 \end{pmatrix}.$$

So the line of best fit is $y = 5x - 10645$. So the estimate for the number of people who will apply in 2018 is $5 \cdot 2018 - 10645 = 93.3$.

2.3 Problems

16. **TRUE** False The matrix A^TA will always be square.

Solution: If A is $m \times n$, then A^T is $n \times m$ so A^TA is $(n \times m)(m \times n) = n \times n$ so it will be square.

17. Consider the set of points $\{(-2,-1),(1,1),(3,2)\}$. Calculate the line of best fit.

Solution: We first write it in matrix form as

$$\begin{pmatrix} -2 & 1 \\ 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

Then we calculate $A^TA = \begin{pmatrix} 14 & 2 \\ 2 & 3 \end{pmatrix}$ and $A^T\vec{b} = \begin{pmatrix} 9 \\ 2 \end{pmatrix}$. Then

$$\begin{pmatrix} a \\ b \end{pmatrix} = (A^TA)^{-1}A^T\vec{b} = \frac{1}{42-4} \begin{pmatrix} 3 & -2 \\ -2 & 14 \end{pmatrix} \begin{pmatrix} 9 \\ 2 \end{pmatrix} = \begin{pmatrix} 23/38 \\ 10/38 \end{pmatrix}.$$

So the line of best fit is $y = 23/38x + 5/19$.

18. Find the line of best fit and the error of the fit of the points $\{(-1,2),(0,-1),(1,1),(3,2)\}$ and use it to estimate the value at 2.

Solution: We first write it in matrix form as

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 2 \end{pmatrix}.$$

Then we calculate $A^T A = \begin{pmatrix} 11 & 3 \\ 3 & 4 \end{pmatrix}$ and $A^T\vec{b} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$. Then

$$\begin{pmatrix} a \\ b \end{pmatrix} = (A^T A)^{-1} A^T \vec{b} = \frac{1}{44-9} \begin{pmatrix} 4 & -3 \\ -3 & 11 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 8/35 \\ 29/35 \end{pmatrix}.$$

So the line of best fit is $y = 8/35x + 29/35$. Thus, $y(2) = 8/35(2) + 29/35 = (16+29)/35 = 45/35 = 9/7$.

19. Consider the set of points $\{(-2,-1),(1,1),(3,2)\}$. Calculate the square error if we estimate it using the line $y = x$. Then calculate the square error if we use the line $y = 0$. Which is a better approximation?

Solution: Using the fit $y = x$ gives the error $(-1-(-2))^2 + (1-1)^2 + (2-3)^2 = 1+0+1 = 2$. Using the approximation $y = 0$ gives the error $(-1-0)^2 + (1-0)^2 + (2-0)^2 = 1+1+4 = 6$. Thus $y = x$ is the better fit.